Listening to Geometry

Studying geometric transformations in music—a real-world application of mathematics—deepens students’ understanding of the underlying principles.
The many connections between music and mathematics are well known. The length of a plucked string determines its tone, the time signature of a piece of music is a ratio, and note durations are measured in fractions. One connection commonly overlooked is that between music and geometry—specifically, geometric transformations, including translations, rotations, reflections, and dilations. Composers have been using these transformations for centuries. J. S. Bach used translations to structure his fugues, Johann Pachelbel included reflections in his *Canon in D Major*, Igor Stravinsky incorporated rotations in *Epitaphium*, and the contemporary composer Charles Wuorinen has used dilations.

The music connection holds great potential for the high school geometry classroom. High school students’ passion for music can be a great motivational tool. The music connection can easily be
personalized to individual student tastes, and it provides an application to life as well as a link to another discipline.

The Connections Standard was one of the original Process Standards identified in NCTM’s Curriculum and Evaluation Standards (1989). In NCTM’s revised Standards document, Principles and Standards for School Mathematics (2000), the Representation Standard joined the Connections Standard as an important part of the pre-K–12 curriculum. Together, these two standards encourage teachers to apply mathematics to contexts outside mathematics and “to model and interpret physical, social and mathematical phenomena” (p. 70).

This article will show how composers use geometric transformations in their music and will provide some activities that you can use in your classroom. If you are uncomfortable with some of the music theory we describe, help is just down the hall—you can collaborate directly with your school’s music teachers.

BASIC MUSIC THEORY AND ITS MATHEMATICAL INTERPRETATIONS
The geometric transformations we will discuss here include translations, reflections, rotations, and dilations.

Translations
A geometric translation can be thought of as sliding an object from one place to another without changing its orientation (Usiskin et al. 2003, p. 302). Musicians use translations in two basic ways.

The first type of musical translation is the horizontal translation—shifting a melody, chord, or other musical object to a later time. Think of the children’s song “Row, Row, Row Your Boat.” One person starts singing the melody, and then, two measures later, a second person begins singing the melody. Any time a melody line repeats in music, a translation in time has occurred. Singing this musical round and others might be a nice way to introduce this application of geometric translations.

In addition to rounds, refrains are an example of horizontal translations. Have your students think of songs that have refrains, such as the Mamas and the Papas’s 1960s hit “California Dreamin’.” Every time the refrain occurs, a translation occurs. Have your students name as many songs as possible where lines or parts of lines are repeated. One example would be the words to the common birthday song, where the second and fourth lines are time translations of the first line. The melody for those lines, however, is not a perfect musical translation.

The second type of musical translation is the vertical translation. This type involves moving a melody line or chord up or down the staff. Musicians call this technique transposition. As listeners,
we often hear it as a key change. **Figure 1a** shows a basic melody line (the French children’s song “Frère Jacques”), and **figure 1b** shows the same melody line transposed to a new key.

The contour of the melody line remains the same; it has just been moved up higher on the staff. Even if you cannot read music, you can see the translation. In your class, you could note that when males and females sing in unison (that is, when they sing the same notes at the same time), they are really performing a translation. If the melody is written in the treble clef (as the melodies in **figs. 1a** and **1b** are), the males are actually singing it an octave lower; they are performing a translation automatically.

**Reflections**

**Reflections** are another type of transformation also commonly used by composers. Mathematically, a reflection over line $m$ is a transformation whereby any point $P$ not on $m$ is mapped to a point $P'$ such that $m$ is the perpendicular bisector of $PP'$. If $P$ is on $m$, then $P$ is mapped onto itself (Usiskin et al. 2003, p. 313). Composers reflect melody lines over both vertical and horizontal lines.

If a melody is reflected over a vertical line—for example, the bar at the end of a measure—the melody is said to be put into **retrograde**. The result is that the original melody is played backward (Mitchell 2004, p. 18). This type of translation is demonstrated in **figures 2a** and **2b**.

You might take the first phrase of your school’s fight song, write out the melody for your students, and then have them reflect the melody horizontally across the last bar line. If you are uncertain about how to do this, work with your school’s music teacher, who could write out the initial phrase for you on musical staff paper. Be sure to let your students listen to this familiar song played backward. At this point, engage the help of instrumental students in your class; they could bring their instruments and play the melody forward and then backward for all to hear.

If a melody is reflected over a horizontal line (or space), such as the initial note F in “Frère Jacques,” the result is called an **inversion**. If retrograde is playing a melody backward, then inversion is playing a melody upside down (Kennedy 1985, p. 353). Which note to invert over is an arbitrary choice, but musically some notes—like do or sol, the tonic (the first) and the dominant (the fifth) notes of the key in which the music is written—would be more recognizable. When you simply turn the music upside down, you are reflecting, or inverting, on B, the note on the third line of the treble clef. Ask your students to explain why. (The third line is the middle of the staff, and it remains the third line whether the music is positioned right side up or upside down.)

Here is an obvious place to incorporate differentiated instruction in your class. Students who are musical or who understand reflections well could reflect, or invert, the melody on a line or space other than the middle line. In particular, have some of them reflect on the tonic, which is the “key” note of the piece (often the first and last note of a piece of music); others could reflect on the dominant, the note that is a fifth above the tonic.

For example, the first note of “Row, Row, Row Your Boat” is the tonic note of that key. If your version of this song is in the key of F, then the fifth of F is C, which is written two spaces above F on the staff. (Why this is a fifth is explained later in this article.) As before, ask the musicians in your class to bring their instruments so that everyone can hear the inversions—first, the simple reflection on the middle line of the staff, then the reflection on the tonic note, and finally the reflection on the dominant (the fifth) note.

Up to this point, we have been able to avoid the need for **accidentals**, notes not in the key signature (sharps, flats, and naturals). **Figures 3a** and **3b** show the basic melody line and the inversion of the melody without accidentals. You can see that the reflection is on the initial note F.

This reflection is not the true inversion of the basic melody because it does not preserve distance mathematically or musically. In music, the distance between two notes is described as an interval and is often subdivided into two parts. For the purposes of the geometry class, we can simplify this process.

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**Fig. 1** A melody (“Frère Jacques”) in F major (a); the basic melody transposed up to C major (b)

**Fig. 2** The basic melody (a); the retrograde of the basic melody (b)
Composers often give written directions in their music to tell the performer to speed up (accelerando) or slow down (ritardando). In these cases, the exact dilations or contractions are left to the interpretation of the musician performing the piece. However, sometimes composers write the melody so that it is played twice as fast or twice as slow. Figure 7 shows our basic melody written so that it will be played twice as fast as the original.

One other type of dilation was used by certain twentieth-century composers, notably Anton Webern, Alban Berg, and Arnold Schoenberg, who used mathematics in their work. They would assign each note a numeric value and then multiply all the values by the same number to get a new set of notes. In addition, these composers placed an artificial constraint on themselves. Traditional music has a tonal center, or key, which is a reference point for the melody to move away from and return to for resolution. In their music incorporating their method of “composing with twelve tones only related to one another” (Schoenberg’s term), these composers would not repeat a note until all of the eleven other notes had been played, an approach designed to keep one note from being more important than any other note. As a result, they created a form called atonal music.

Listening to atonal music can be a surprising experience. You might want to bring in samples of tonal and atonal music (check with your school’s music teacher) and play them. Your students will easily hear the difference.

The artificial constraint of not allowing repeated notes has an interesting mathematical consequence. There are twelve tones in Western music. The atonal composers assigned to C the value zero, to C# the value 1, to D the value 2, and so on up the scale to B, which was assigned the value 11. When dilating a “melody” line, these composers would look at the numeric values and multiply them by some scalar. However, because of the constraint, composers could multiply only by 5, 7, and 11. Why?

### Rotations

Rotations are transformations that preserve distance and have exactly one fixed point. Rotations are also compositions (in the mathematical sense) of reflections (Usiskin et al. 2003, p. 315). One musical rotation—retrograde inversion—composes the two types of reflections discussed above. It is simply the retrograde of the inversion: in other words, flip the melody upside down and then play it backward (Mitchell 2004, p. 19). Figure 6 (a, b, and c) shows the process. The retrograde inversion is essentially a 180-degree rotation about the first note of the melody.

### Dilations

The last geometric transformation we will consider is dilation. Dilations are created by changing the size of an object by a scalar multiple. If the scalar is between –1 and 1, the transformation reduces the size of the object and may be called a contraction.

This concept is used in music in two ways. First, dilations are used in music to change time durations. Composers often give written directions in their music to tell the performer to speed up (accelerando) or slow down (ritardando). In these cases, the exact dilations or contractions are left to the interpretation of the musician performing the piece. However, sometimes composers write the melody so that it is played twice as fast or twice as slow.
The answer is that these are the only values in the system relatively prime to 12, meaning that they were the only numbers that would preserve the atonality. (Technically, 1 is also relatively prime to 12, but multiplying by 1 would simply repeat the given melody. This process would be the identity transformation.) In practice, composers would often translate the resulting dilations up or down octaves to keep the tones within an acceptable and audible range. Otherwise, multiplying by 11 could result in a transformed melody with a twenty-octave range (and no instrument has that extensive a range).

Consider the C-major chord (C, E, G). This chord would be denoted numerically as (0, 4, 7). Multiply by 5 (0, 20, 35) and reduce modulo 12 to get a new chord (0, 8, 11), or (C, G♯, B). However, if you multiply by 3 (a factor of 12), you get (0, 12, 21), which, when reduced modulo 12, gives (0, 0, 9) or (C, C, A). Multiplying by 3 produced a repeated note, unwelcome in these composers’ atonal scheme.

So, go back to your school’s fight song and have your students “write” the song numerically by using the numbers 0 through 11, as described here. Then have different groups multiply by 2, 3, 4, or 5 to get a new numerical representation of the song. However, before students can convert the fight song back into musical notes, they have to “reduce” the numbers that are larger than 11. They do this by dividing the new number by 12 and assigning the value of the remainder to the note. (This process is what was meant by reducing modulo 12.) Once the groups have finished this step, they can rewrite the school’s fight song in its dilated and octave-reduced form and play it on an instrument to hear the results.

Because many melodies have a range larger than an octave, you might need to expand this exercise to 24 (instead of 12) to encompass a two-octave range. The second octave C would be 12, C♯ would be 13, D would be 14, and so on; the final B would be 23. This expansion would provide some additional prime numbers to be used for the scalar multiple, and the final result would have to be reduced modulo 24, rather than modulo 12.

As the students compose, have them label each transformation. Using solely translations and reflections will result in a piece of music that is pleasing to the ear.
CULMINATING PROJECT AND OTHER IDEAS

Now that we have seen how geometric transformations occur in music, we can combine them in the classroom. The following project could be the culmination of a unit on geometric transformations. When we did this with a group of students, they came up with additional examples of transformations that we had not identified. We have included the students’ ideas here.

Begin by having students find a simple melody line that will become the basis for a new composition. They may use an existing melody that they like (such as their school’s fight song) or write their own melody, or you might allow the class a few minutes to write a melody as a group. This melody should not be too long, just a few measures (four is a good length). The assignment will be to compose a piece of music, based on a common basic melody, of a given length. If students use a four-measure melody, thirty-two measures is a good length for the final result and will help ensure that students use a variety of transformations.

As the students compose, have them label each transformation. Using solely translations and reflections will result in a piece of music that is pleasing to the ear, but if the students do nothing more than translate and reflect, the resulting compositions will all sound similar.

Dilations allow students to show real creativity. Speeding up and slowing down the music in apt places can help a composition sound less mechanical. In addition, dilations can be applied to the duration of individual notes within the melody, allowing students to transform the rhythm as well as the melodic contour. Two samples of student work are shown (see figs. 8 and 9). The students were given the first four measures of Bach’s “Little” Fugue in G Minor, written for organ, along with the piece’s final measure (see fig. 10) to help them resolve their composition nicely.

In doing this activity, our students found additional ways of interpreting geometric transformations in music, and yours probably will as well. For example, they suggested that a volume, or dynamic, change could be a translation—a louder section is interpreted as translating up, and a softer section is recognized as translating down.

Our students noticed several additional uses and interpretations of the transformations. For example, changing clefs (from treble clef to bass clef and vice versa) is a musical transposition as well as a geometric translation. They also noted two new uses of dilations and contractions. Musically, a trill or tremolo (two notes that repeat rapidly) is a contraction in time with a translation in time, and inverting the time signature also results in a time dilation or contraction. For example, the time signature 2/4 means that there are two beats to a measure and that the quarter note is equal to one beat; the reciprocal time signature, 4/2, means that there are four beats to a measure and that the half note is equal to one beat. This change doubles the pace of the music while extending the length of each measure.

A FINAL THOUGHT

You can use these geometric transformations in music as a “dynamic opening” (Barger 2006) for your class. While teaching a unit on geometric transformations, you could start each class with a clip of music that uses the type of transformation the students will be studying that day. For example, Bach’s “Little” Fugue in G Minor is full of
translations. It begins with a melody, or theme, in an upper voice. After the melody is completed, the upper voice begins a second melody, or countertheme, but the original melody is sounded by a lower voice. This example reveals a transposition—a vertical translation (i.e., playing the melody in a lower voice)—as well as a translation in time—a horizontal translation (i.e., repeating the melody).

Finally, encourage students to bring in examples of music in which they hear geometry. While playing a clip of the music, they can explain the transformations they hear. Imagine beginning each class by listening to geometry.

REFERENCES

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